First name ....   Date ....

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Exercise 410

Determination of Young's modulus by the beam-bending method

Beam size measurements

Beam type	lenght*	width	thickness	C
	l, [m]	a_i, [mm]	h_i, [mm]	[m^{-1}]

* **Note**: The bar length is calculated between the support points.

Measurements of the depression of a beam

Beam type	Load weight	Micrometer readings, [mm], under load		Depression of a beam Y_i	Q_i/Y_i	Mean, $k = Q/Y$	Young's modulus, E	**Theoretical** Young's modulus, E
	[g]	increasing	decreasing	[mm]	[N/m]	[N/m]	[GPa]	[GPa]
	0			-----	-----			
	0			-----	-----			

Exercise 410. Determination of Young's modulus by the beam-bending method

Purpose

The aim of this exercise is to determine the Young's modulus of metals and wood by the flat beam bending method by measuring the depression of a beam.

Theory

Hook's law

If we apply force to an immobilized elastic body, stresses will arise in this body, causing its deformation. *The stress* σ in a beam with a cross-section *A*, on which a force $\vec{F}$ acts (perpendicular or tangent to *A*) is equal to the ratio of the force to the cross-sectional area of the beam:

$$\sigma = F/A \tag{1}$$

Stress resists the intermolecular forces inside the material. There are usually three types of stress: stretching (lengthening the body), compressive (shortening the body) and shear (deforming the shape of the body). In the latter case, the force acts tangent to the cross-sectional area.

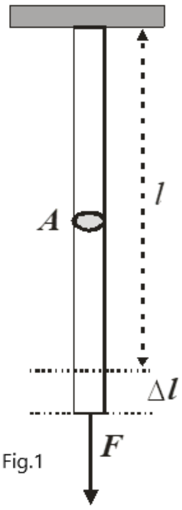


Fig.1

The change in beam length due to tension or compression is proportional to its length. If, for example, beam in lenght *l*, we stretch by force $\vec{F}$, it increases its lenght by Δ*l*, rys. 1, then a measure of deformation ε is the relative change in length:

$$\varepsilon = \Delta l / l\,. \tag{2}$$

When after removing the force $\vec{F}$ the body returns to its dimensions, this *deformation is called elastic.* With small deformations, *ε* is proportional to σ:

$$\varepsilon = \frac{1}{E} \cdot \sigma\,. \tag{3}$$

E is *modulus of elasticity* (called *Young's modulus*) of a material. Young's modulus is numerically equal to the stress at which the relative change in beam length would be equal to one. Young's modulus is expressed, like stress or pressure, in pascals: $1\ \mathrm{Pa} = 1\ \mathrm{N/m^2}$.

The linear relationship between stress and strain is known as the *Hooke's law*. After substituting into (3) the formulas defining ε and *σ*, we get :

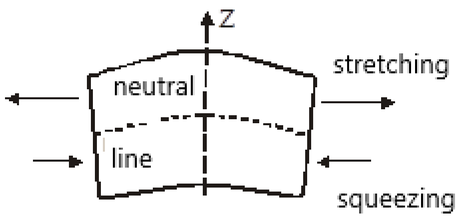
$$\Delta l = \frac{1}{E} \cdot \frac{l}{A} F \,. \qquad (4)$$

Thus, Hooke's law states that when stretching or compressing, the change in length is proportional to the force acting.

The easiest way to determine the Young's modulus is to measure the increment in length Δl of the rod in lenght l and cross-sectional area A, fixed at one end and stretched by force F. However, in the case of thicker beams, it is difficult to obtain measurable elongations due to the need to use very high forces. For this reason, we use complex deformations, which include bending a beam fixed on one side or supported on both ends.

Bending of the beam

Bending of the beam can be reduced to its simultaneous stretching and compression. There is a layer along the bent beam, called *neutral surface*, the length of which does not change when bent. Above this surface, the deformation forces take the tensile direction of the upper layers, below - the opposite direction and cause compression of the lower layers.



fragment of the cross section of the beam

These forces occur in pairs and create the bending moment $\vec{M}$ to the neutral line.

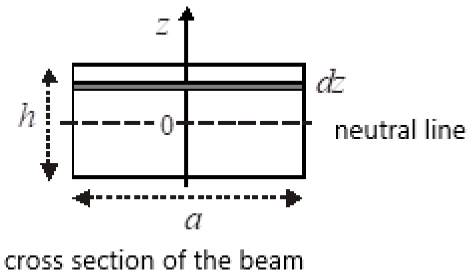
The following relationship between the bending moment and the modulus of elasticity of the beam can be derived:

$$M = \frac{E\,I}{R} \,. \qquad (5)$$

In this formula R is the *radius of curvature* of the deflected beam (it is the radius of the circle whose fragment is the deflected beam), while I denotes the *moment of inertia of the section*.

The moment of inertia of the cross-section is determined by the distribution of the elements of the cross-sectional area with respect to the neutral line. If by z we express the distance of the cross-sectional area element dS from the neutral line, then I is defined by the formula:

$$I = \int_S z^2 dS$$



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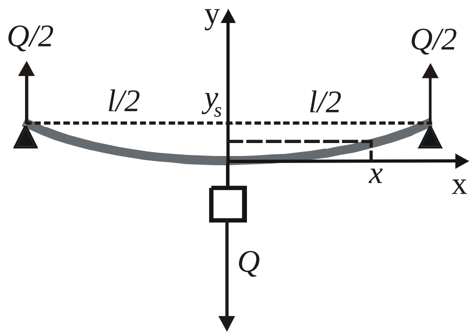
Calculating this area integral for a rectangular section of width *a* and thickness *h* we obtain the following formula for the moment of inertia of the section:

$$I = \frac{a \cdot h^3}{12}. \tag{6}$$

We consider the deflection of a beam of length *l* supported at both ends and loaded with a mass *m* in the center by a weight *Q*. Each support acts on the beam with a reaction force equal to $Q/2$, and the central part of the beam remains horizontal. We will consider the deflection of the beam in relation to the coordinate system whose origin is located in the middle of the beam. The moment of the reaction force acting on the end of the beam, calculated in relation to the point lying at the distance *x* from the center of the beam, is (with small deflections):

$$M = \frac{Q}{2} \cdot \left(\frac{l}{2} - x \right).$$

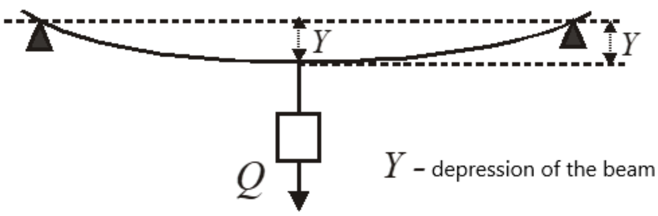


The radius of curvature *R* of the deflected beam is given by the equation, the approximate form of which, in our case, is as follows:

$$\frac{1}{R} = \frac{d^2 y}{dx^2}.$$

After substituting the last two formulas in relation (5), we will obtain an equation whose solution determines the beam deflection line $y = f(x)$.

If in the function $y = f(x)$ we substitute *x* the value of the coordinate at the fulcrum, $x = l/2$, we obtain the maximum value of the coordinate $y = y_{\max} = Y$.



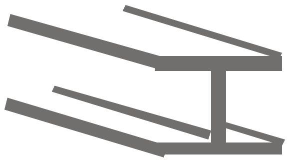
The value of the *y* coordinate in the place of support is called the *depression of a beam Y*. The formula for the depression of a deflected beam is as follows:

$$Y = \frac{Q}{48EI} l^3.$$

For a rectangular section, we get:

$$Y = \frac{Q l^3}{4Eah^3} \qquad (7)$$

The formula for the depression of the beam shows that the deflection of the beam is inversely proportional to the moment of inertia of the section and if the beam has a rectangular section, then the depression is inversely proportional to the thickness of the beam *h* raised up to the third power.



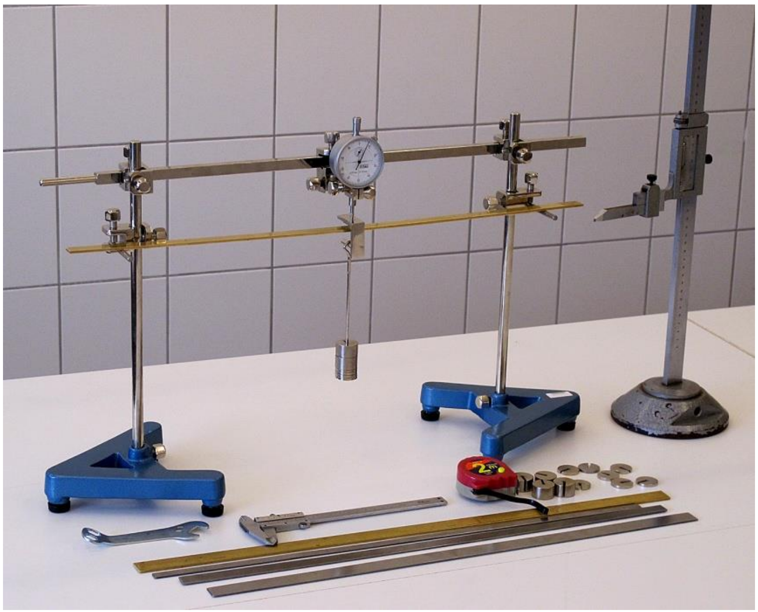
The above conclusions suggest that in order to construct strong, lightweight elements, most of the material should be located as far away from the neutral surface as possible.

For example, an I-section resists better moments of bending forces acting perpendicular to its length than a beam with a square cross-section made of the same amount of material.

Measurement of the depression of the beam *Y* for a given load Q allows to determine the Young's modulus of the material from which the bar is made. By transforming the formula (7) we get:

$$E = \frac{l^3}{4ah^3} \cdot \frac{Q}{Y} \,. \qquad (8)$$

REQUIRED EQUIPMENT	• Flat metal and wooden beams
• Two tripods with a set of clamps	• Caliper
• Crossbeam	• Measuring tape
• Dial micrometer sensor	• A set of weights with a mass 10 g and 50 g
• Catetometer for setting the height of the supports supporting the bars	• Stape and hanger for suspending the load



Performance of the task

1. Preperation of the measuring system

- We check whether the measuring system is prepared in accordance with the attached photo.
- We choose two or three beams for measurements. The beams are metal (iron, brass) and wooden.

2. Beam size measurements

- We measure the distance *l* between the supports' centers with a millimeter measure. The result is a measure of the effective length of the bent beam.
- We check the width *a* and thickness *h* of the selected bars with a caliper. The bars can have the following widths, in [mm]: 10; 15; 20 and thickness, in [mm]: 1,5; 2,0;3,0;5,0.
- We calculate the beam constant C: $C = \frac{l^3}{4ah^3}$.

3. Determination of the ratio Q/Y

- We weigh the weight hanger m_w, and check the mass of the attached weights.
- The tested beam is placed on supports attached to the tripods.
- Put a stapes on the middle of the rod and place it under the micrometric sensor so that its tip rests on the recess on the upper surface of the stapes.

  ***Note**: Do not bend the rod to slide it under the sensor tip, only lift the sensor tip with the handle above the dial.*
- We read the indications of the micrometer y_0. This is a zero reading (for a beam loaded with a stapes only).

  ***Note**: the micrometer gauge reading for the stapes bar should be approximately 5 mm when the stapes is on a 2 mm thick bar. If necessary, notify the instructor of the need to adjust the height of the supports.*
- The first load (Q_1) is suspended from the stapes (hanger and weights). The load value should be between 100 g and 150 g. We read the micrometer reading.

  ***Note:** The stape may move when the bar is loaded, therefore it is necessary to check the position of the tip of the micrometer sensor in relation to the stape each time the load is changed.*
- The difference $Y_1 = |y_1 - y_0|$ gives first depression of the beam Y_1.
- Measurements of depressions of the beam $Y_i = |y_i - y_0|$ we carry out two more times, increasing the load each time by 100 g (or according to the teacher's instructions).
- First, we determine the depressions of the beam with increasing and then decreasing loads. From the two results obtained for a given load value, we calculate the average value, which we take as the proper value of the depression of the beam.
- We calculate the quotient for each load Q_i/Y_i, $i = 1, 2, 3$.
- We calculate the average value Q/Y. If we introduce the designation for a single measurement: $k_i \equiv Q_i/Y_i$ and for the mean value $k \equiv Q/Y$, then we can express it as follows:

$$k = \frac{k_1 + k_2 + k_3}{3}, \tag{9}$$

4. Calculating the Young’s modulus

According to the formula (8), the product of the bar constant C and the value of k (the average ratio of Q to Y) is equal to the Young's modulus for a given bar with a rectangular cross-section:

$$E = C \cdot k\,. \tag{11}$$

Error calculus

Measurement error *Δk*, we calculate as the absolute maximum error between the mean value of *k* and each of the three measurements k_i:

$$\Delta k = \max|k - k_i|; \qquad i = 1,\ 2,\ 3.$$

We determine the remaining errors of complex physical quantities using the logarithmic derivative method.

$$\frac{\Delta C}{C} = 3\left(\frac{\Delta l}{l} + \frac{\Delta h}{h}\right) + \frac{\Delta a}{a}\,.$$

We assume: *Δl* = 4mm, *Δa* = 0,05mm, *Δh* = 0,05mm.

$$\Delta E = E\left(\frac{\Delta k}{k} + \frac{\Delta C}{C}\right).$$

We also calculate the relative percentage error of Young's modulus: $B_p = \frac{\Delta E}{E} \cdot 100\%.$

Conclusions:

We compare the calculated values of Young's modulus with the physical table values. We calculate errors with respect to an array value. We compare these errors with the values from the error calculus.