First name .... Date ....

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Exercise 415

Measurement of the mass and mean density of the globe

I. Measurement of the gravitational acceleration

Mass of the weight m, [g]	Pendulum lenght L, [m]	Number of swings n	Time of n swings t, [s]	Period of oscillations $T = t/n$ [s]	Gravitational acceleration g_i, [m/s^2]

II. Results

		Calculated value	*Relative error* [%]	*Reference value**
Gravitational acceleration $\bar{g}$	[m/s^2]			
Mass of Earth M	[kg]			
Mean density of the globe	[kg/m^3]			

* search in Physical and Mathematical Tables

Exercise 415. Measurement of the mass and mean density of the globe

1. Introduction

The aim of the exercise is to measure the mass and mean density of the Earth by measuring the gravitational acceleration by using simple pendulum.

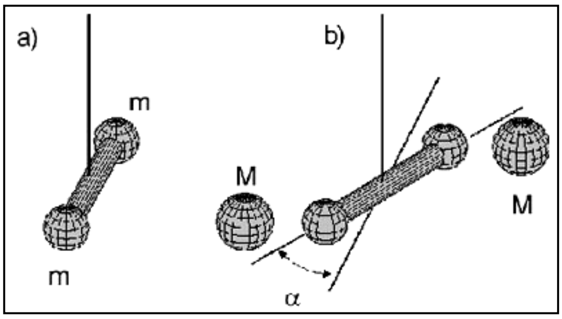
1.1 Gravity

The mutual attraction force F of spherically symmetrical bodies with masses M and m has a value:

$$F = G\frac{Mm}{R^2}, \tag{1}$$

where R is the distance between the centres of masses of the two bodies and G is the gravitational constant.

For the curious
The value of the gravitational constant was determined in 1798 by Henry Cavendish by means of a torsion balance. On a springy quartz thread, Cavendish hung a rod horizontally with two small lead balls with masses m (Figure a). He then placed a larger lead ball of mass M near each ball and measured precisely the angle by which the rod rotated (Figure b). From the measurement of the angle of rotation, he determined G.



The currently accepted value of G is: $G = 6,673 \cdot 10^{-11}\ \mathrm{N \cdot m^2 \cdot kg^{-2}}$

Since we know the Earth`s radius (e.g. from measuring the Earth`s circumfence) and the gravitational acceleration g, the value of the G constant allows us to calculate the Earth`s mass M and the mean density ρ of our planet (4):

$$G\frac{Mm}{R^2} = mg \tag{2}$$

$$M = \frac{gR^2}{G} \tag{3}$$

$$\rho_{Ziemi} = \frac{M}{\frac{4}{3}\pi R^3} \tag{4}$$

Gravitational acceleration depends on latitude (due to the centrifugal force caused by the rotation of the globe and the fact that the globe is slightly flattened at the poles) and altitude (mainly due to the change of distance from the centre of the Earth), but this changes are relatively small. Even between the equator and the pole the difference do not exceed 0.5%. The g value for Warsaw (100 m above the sea level) is 9.81230 m/s^2.

From the geographical measurements, the mean radius of the globe R is 6371 km.

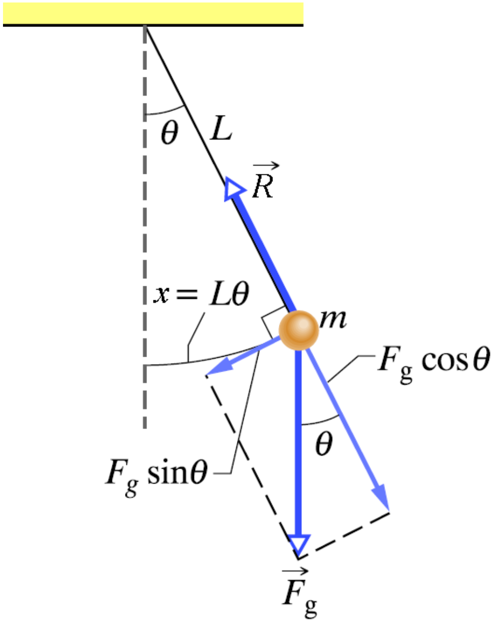
1.2 Mathematical pendulum

A mathematical pendulum is a point-mass m swinging in one plane on an inextensible and massless string of length L.

Labels:

m – pendulum mass, L – pendulum length,

θ – deviation angle from the equilibrium position,



F_g – gravitational force, $F_g = mg$,

t – time, g – gravitational acceleration,

ω – angular frequency, T – period of oscillation,

x – displacement of mass m along the curve, $x = L\theta$,

θ_m – maximum deflection of the pendulum (amplitude of oscillation)

R – string reaction force, $\pi = 3{,}14159...$.

The component of gravitational force $mg\cos\theta$ pulls the string and is balanced by the string reaction force R. The weight is set in motion by the component of gravitational force $mg\sin\theta$. From the second Newton`s law of motion results, that $ma = mg\sin\theta$. Since $a = d^2x/dt^2$ and $x = L\theta$, the equation of weight motion can be presented in the form:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta \quad (5)$$

This equation does not include the mass of the weight, from which it follows that the motion of the pendulum and therefore its period of oscillation does not depend on the mass of the weight.

Equation (5) does not describe the motion of a simple harmonic oscillator, because the second derivative of the deflection angle θ is not proportional to θ but to $\sin\theta$. This means that the period of oscillation of pendulum will depend on the amplitude of oscillation. For small values of the angle (for small pendulum deflections) an approximation can be used and the motion of the weight can be described by the equation of a simple harmonic oscillator:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \quad (6)$$

The angular frequency for such oscillator is expressed by the formula:

$$\omega = \sqrt{\frac{g}{L}} \quad (7)$$

And the period of oscillation:

$$T = 2\pi\sqrt{\frac{L}{g}} \quad (8)$$

The period of oscillation of a pendulum (without approximation of small angle deviation) is obtained by solving the equation (5):

$$T = 2\pi\sqrt{\frac{L}{g}}\cdot\left[1 + \left(\frac{1}{2}\right)^2\sin^2\left(\frac{\theta_m}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\sin^4\left(\frac{\theta_m}{2}\right) + \ldots\right] \quad (9)$$

The results of calculating the period of oscillation according to the full formule (9), summarised in Table I, allow us to assess the error by using the small angle deviation approximation.

Table I.

θ_m (degrees)	**Period error for the small angles deviation approximation**
5	0.048%
10	0.19%
15	0.43%

20	0.75%
30	1.7%
40	3.2%

It can be concluded from the Table I that for pendulums with maximum deflection angles of less than 20 degrees, the use of the simplified formula results in an error smaller than 1%.

By measuring the lenght *L* and period of oscillation *T*, it is possible to calculate the gravitational acceleration *g* from the formula:

$$g = \frac{4\pi^2 L}{T^2}. \tag{10}$$

Having determined the value of the gravitational acceleration, from the equations (3) and (4) we can calculate the mass and the mean density of the Earth, respectively.

2. Performance of the task.

1. Record the lengths and masses of the pendulums.
2. Set the pendulums in motion one after the other with a small amplitude (**the maximum deviation should not exceed 5 cm**) and so that the oscillations take place in one plane, perpendicular to the wall.
3. Measure the time for 20 full oscillations for the long pendulums, and 30 for the short pendulums. Remember that one full oscillation is when the weight returns to the position in which the time measurement was started.
4. Enter the results in the measurement table.
5. Perform the calculations.

3. Calculation of the uncertainties

In the exercise we are dealing with a measurement of *g* repeated 10 times. Let us denote the succesive results of the N times repeated measurement by g_i, where the subscript *i* denotes the measurement number ($i = 1, ..., N$). The arithmetic mean $\bar{g}$ of the measurement results is a good estimation of the „true" value of *g:*

$$\bar{g} = \frac{1}{N}\sum_{i=1}^{N} g_i = \frac{g_1 + g_2 + ... + g_N}{N} \tag{11}$$

For the measurement uncertainty of the arithmetic mean of the *g* value, we take the root mean square error (so called standard deviation of the mean value):

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}\left(\bar{g} - g_i\right)^2}{N\left(N-1\right)}} \tag{12}$$

We consider that the error of the determined *g* value is σ, so obtained value can be written as $g = \bar{g} \pm \sigma$.

The relative error of determining the mass and mean density of the Earth is equal to the relative error Δg/g:

$$\Delta g / g = \sigma / \bar{g}.$$

In the conclusions, in addition to comparing the results obtained with reference values, consider whether:

- the mass of the weight influences obtained results;
- the mathematical pendulum approximation is valid for the pendulums used;
- were there other factors not considered in the analysis of the pendulum movement.